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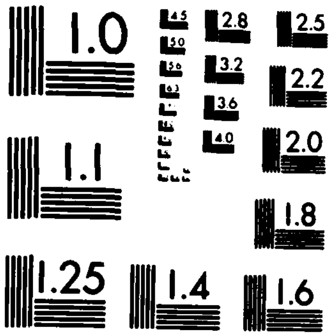

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PERFORMANCE OF THE h, p and h-p VERSIONS OF THE FINITE ELEMENT METHOD

by

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The paper summarizes the main characteristic behaviors of the three basic versions of the finite element method. The sample theorems are formulated and numerical examples illustrate the results. The paper is the content of the invited lecture at the Symposium on Advances and Trends in Structures and Dynamics, Arlington, Virginia, October 1984.

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PERFORMANCE OF THE h, p and h-p VERSIONS OF THE FINITE ELEMENT METHOD

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1. INTRODUCTION

There are three basic versions of the finite element méthod, called the h, p and h-p versions. They are essentially characterized by the way in which the finite element meshes and polynomial degree of elements are chosen. They differ in computer implementation (program architecture) and mathematical analysis. This paper is concerned mainly with the question of how the meshes and polynomial degree of the elements affect the accuracy of finite element solutions. Our approach is to fix certain parameters or their relation and increase the number of degrees of freedom so that the finite element solutions converge to the exact solution. Such a systematic increase of the number of degrees of freedom is called extension because it can be interpreted as a systematic extension of finite element spaces.

When emphasis is on analysis of accuracy and not aspects of implementation, then we speak about the h, p and h-p extensions rather than versions. Understanding the various extension processes and their numerical performance is essential for resolving certain basic questions of implementation.

The h-extension is the most commonly used approach to error reduction. The polynomial degree (p) of the elements is fixed and the errors of approximation are reduced through mesh refinement. The size of the elements is usually denoted by h, hence the name: h-extension. Typically, the polynomial degree of elements is low, usually $p = 1$ or $p = 2$.

In the p-extension the mesh is fixed and convergence is achieved by increasing the polynomial degree of elements either uniformly or selectively.

The h-p extension combines the h- and p-versions, i.e., reduction of error is achieved by mesh refinement and concurrent choices in the polynomial degree of elements.

The parameters that characterize extension processes can be chosen either *a*

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priori, on the basis of certain characteristics of the exact solution, known a priori, or a posteriori, through utilization of some feedback procedure in which case the parameters of the extension process depend on previously computed data.

The analysis and especially optimization of extension processes (selection of optimal meshes, polynomial degree distributions, etc.) presented herein indicate the potential of alternative approaches and provide a basis for decisions concerning implementation.

In order to keep the essential points in focus, we consider only two simple model problems based on the displacement formulation and two measures of error: The error measured in energy norm and the error of stress components computed at specific points.

Specifically, we denote the exact and finite element solutions respectively by u_0 and $\tilde{u}$. The error is then $e = u_0 - \tilde{u}$. The energy norm of e is denoted by $\|e\|_E$ and is defined as the square root of the energy of the error:

$$\|e\|_E = (W(e))^{1/2}.$$

The relative error in energy norm is denoted by $\|e\|_{ER}$ and is defined as:

$$\|e\|_{ER} = \frac{\|e\|_E}{\|u_0\|_E}.$$

The error in stress components at some poin x_0 is defined as

$$e_{ij}(x_0) = |\sigma^0_{ij}(x_0) - \hat{\sigma}_{ij}(x_0)|$$

where $\sigma^0_{ij}(x_0)$ and $\hat{\sigma}_{ij}(x_0)$ respectively denote the exact and computed components of the stress tensor at point x_0. The relative error in stresses is defined by:

$$e^R_{ij}(x_0) = \frac{|e_{ij}(x_0)|}{|\sigma^0_{ij}(x_0)|}$$

The one dimensional problem can be analyzed theoretically and experimentally in great detail. One dimensional problems can also serve as models for higher dimensional problems which are vastly more complicated and less well understood. Presentation of details and derivation of formulas quoted herein is beyond the scope of this paper. For the proofs of theorems in one dimension, we refer to [1], [2]. For further details and application to two dimensions, we refer to [3], [4], [5], [6], [8].

2. MODEL PROBLEMS

2.1. Model Problems

We consider the following simple model problems:

$$-u''(x) = f(x), \qquad x \in I = (0,1) \tag{2.1}$$

$$u(0) = u(1) = 0 \tag{2.2}$$

and exact solutions of the form:

$$u_0(x) := (x+\xi)^\alpha - \xi^\alpha - x[(1+\xi)^\alpha - \xi^\alpha] \tag{2.3}$$

with $\alpha > 1/2$ and $\xi > 0$. The solutions minimize the potential energy defined as:

$$\pi(u) = W(u) - 2\int_0^1 fu\,dx$$

where $W(u) = \int_0^1 (u')^2\,dx$.

The finite element solutions are characterized by the mesh and p-distributions. Specifically we denote the mesh by the partition:

$$\Delta =: 0 = x_0^\Delta < x_1^\Delta < \cdots < x_{M(\Delta)}^\Delta = 1$$

where $x_0^\Delta, x_1^\Delta, \cdots, x_{M(\Delta)}^\Delta$ are the mesh (or nodal) points. The j^{th} finite element is denoted by: $I_{j+1} = (x_j^\Delta, x_{j+1}^\Delta)$. The size of the j^{th} element is defined by $h_j = x_j^\Delta - x_{j-1}^\Delta$. The size of the largest element is denoted $h(\Delta)$. The set of all functions w, defined on I that satisfy the following conditions:

(a) $W(w) < \infty$,

(b) the boundary conditions (2.2),

(c) on I_j^Δ w is a polynomial of degree p_j^Δ,

is denoted by $S^{\underline{p}(\Delta)}(\Delta)$ where $\underline{p}(\Delta) = \left(p_1^\Delta, p_2^\Delta, \ldots, p_{M(\Delta)}^\Delta\right)$ is the vector of p-distribution. $S^{\underline{p}(\Delta)}(\Delta)$ is called the finite element space. The number of degrees of freedom is denoted by $N(\Delta)$:

$$N(\Delta) = \dim S^{\underline{p}(\Delta)}(\Delta) = \sum_{j=1}^{M(\Delta)} p_j^\Delta - 1.$$

The finite element method consists of finding $\hat{u} \in S^{\underline{p}(\Delta)}$ which minimizes $\pi(u)$ over $S^{\underline{p}(\Delta)}(\Delta)$. The extension processes are characterized by the selection of sequences of Δ and $\underline{p}(\Delta)$.

2.2. The Two-Dimensional Model Problem

We shall consider the model problem of Fig. 2.1 assuming plane strain conditions and using the elastic parameters $E = 1$, $\nu = 0.3$. The tractions on the boundary are chosen so that the exact solution is known and the singularity at the crack tip is characterized by the stress intensity factors $K_I = K_{II} = 1$.

The solution of Model Problem 1 has certain similarities with the solution of Model Problem 2 from the point of view of numerical performance when $\xi = 0$ and $0.5 < \alpha < 1$.

3. THE PERFORMANCE OF THE FINITE ELEMENT METHOD IN ONE DIMENSION WITH RESPECT TO THE ENERGY NORM MEASURE

We now demonstrate how the performance of the various extension processes depends on the mesh and the polynomial degree of elements. The principal mathematical tool is asymptotic analysis which provides information on how the error depends on the mumber of degress of freedom when only one parameter is being varied, and on certain essential characteristics of the exact solution provided that the number of degrees of freedom is large. We shall consider Model Problem 1 and choose the parameters ξ and α so as to represent problems with solutions of various smoothness ($\xi = 0$, $\alpha > 1/2$) and $\xi > 0$.

3.1. H-Extensions Based on Uniform Mesh Refinement ($\xi = 0$)

In this case $M(\Delta) = \frac{1}{h(\Delta)}$, p is fixed. The number of elements fully characterizes the mesh and the error in energy norm for $\xi = 0$ is estimated as follows:

$$\|e\|_{ER} \mathrel{\hat{=}} \frac{C(\alpha)}{M^{\mu}p^{\rho}} \tag{3.1}$$

where the symbol $\hat{=}$ means "asymptotically equal", $C(\alpha)$ is a constant independent of the mesh and the polynomial degree of elements:

$$\mu = \min(\alpha - 1/2, p) \tag{3.2}$$

$$\rho = 2\alpha - 1 \tag{3.3}$$

and the number of degrees of freedom is given by:

$$N = Mp-1. \tag{3.4}$$

Although (3.1) is an asymptotic estimate, it holds even for reasonably low values of N. In order to demonstrate this, we have computed the value

$$D = \|e\|_{ER} M^{\mu}p^{\rho}. \tag{3.5}$$

D is called the numerical value of $C(\alpha)$. The results for $\alpha = 0.7$ (and therefore $\mu = 0.2$, $\rho = 0.4$) are shown in Table 3.1. $\|e\|_{ER}$ is shown as "percent relative error". It is seen that (3.1) holds well also when $M(\Delta)$ is small and the error is large. In view of the fact that ρ is twice the value of μ, for the same number of degrees of freedom the higher order elements perform better.

3.2. H-Extensions Based on Nearly Optimal Mesh Refinement ($\xi = 0$)

We once again consider the case $\xi = 0$. In this case the optimal mesh (for fixed p) is asymptotically:

$$x_j^\Delta = \left(\frac{j}{M}\right)^{\beta_0} \tag{3.6}$$

where:

$$\beta_0 = \frac{p + 1/2}{\alpha - 1/2} . \tag{3.7}$$

The estimate of error measured in energy norm is given by:

$$\|e\|_{ER} \cong C(\alpha) \frac{\sqrt{\beta}}{p^\alpha} \left(\frac{\beta}{4M}\right)^p \sqrt{\frac{1-M^{-[(2\alpha-1)\beta-2p]}}{(2\alpha-1)\beta-2p}} = \hat{C}\, N^{-p} \tag{3.8a}$$

provided that

$$\beta > \bar{\beta} = \frac{p}{\alpha - 1/2} . \tag{3.8b}$$

When $\beta < \bar{\beta}$ then the rate of convergence with respect to M decreases. The reliability of estimate (3.8) for $\beta = \beta_0$ is shown in Tables 3.2a and 3.2b.

With reference to Tables 3.2a, 3.2b, we note the following observations:

(a) The asymptotic estimate (3.8) is of good quality when $M > 2p$. For $M < 2p$ the formula is pessimistic. The reason for this is that (3.8) is based on the assumption that $M \to \infty$. Therefore it cannot be expected to give close estimates for low values of M. In the case $M < 2p$, analysis based on p-extension rather than h-extension should be used.

Table 3.3 shows the error for optimal distribution of the nodal points for $M = 2$ and $\alpha = 1.1$. It is seen that for small values of p ($p = 1,2$) the asymptotically optimal mesh performs very nearly as well as the optimal mesh.

(b) When α is small (strong stress singularity occurs) the optimal refinement is so strong that roundoff limitations are encountered even when the computations are performed in double precision. Table 3.4 shows the values of β_0, $\bar{\beta}$, the coordinate of the first nodal point for the optimal mesh, x_1^Δ (opt), and for the mesh at which the rate of convergence begins to decrease, x_1^Δ (min), for $\alpha = 0.7$, $M = 32$.

(c) Overrefinement is more advantageous than underrefinement. If α is not known precisely, then the refinement should be designed for lower bound estimates of α. Overrefinement increases the value of $\hat{C}$ in (3.8a) but does not alter the rate of convergence (N^{-p}). The penalty, in terms of increased values of C and increased values of N for achieving comparable levels of accuracy are shown in

Table 3.5. The mesh refinement is optimal for $\alpha = 0.7$, the penalty values are shown for $\alpha = 1.1$, 1.6, 2.1.

It is seen that the penalty for overrefinement is not small. Underrefinement also reduces the rate of convergence. Therefore the penalty for underrefinement is still larger. This shows the importance of correct selection of mesh refinement.

3.3. P-Extensions Based on Uniform Meshes ($\xi = 0$)

In this case $M(\Delta) \ll p$ and the estimate (3.1) holds. The results of computational experiments are shown in Table 3.6.

On comparing the results on the basis of the number of degrees of freedom ($N = Mp-1$), it is seen that the best choice is $M = 1$. Comparing Tables 3.6 and 3.2a it is evident that h-extensions based on optimal meshes yield better results than p-extensions based on uniform meshes. The performance of p-extensions cannot be improved substantially through optimizing the p-distribution. Table 3.7 shows the effect of optimal p-distributions for $M = 2$ and $\alpha = 0.7$, 1.1.

3.4. H-p Extensions ($\xi = 0$)

In this case the meshes and p distributions are optimized concurrently. The assymptotically optimal mesh is characterized by the following geometric progression of nodal points:

$$x_j^{\Delta} = \kappa_0^{M(\Delta)-j} \qquad j = 1,2,\ldots,M(\Delta) \tag{3.9}$$

where:

$$\kappa_0 = (\sqrt{2} - 1)^2. \tag{3.10}$$

The polynomial degrees of elements are assigned as follows:

$$p_j = [s(\alpha)j] \tag{3.11}$$

where

$$\underset{\sim}{s}(\alpha) = 2(\alpha - 1/2) \tag{3.12}$$

and $[\cdot]$ means: the integral part.

The error estimate is:

$$\|e\|_{ER} = C(\alpha)\left[(\sqrt{2} + 1)^2\right]^{-\sqrt{(\alpha - 1/2)N}}$$

$$= C(\alpha)e^{-\gamma_0\sqrt{(\alpha - 1/2)N}}, \qquad \gamma_0 = 1.574. \tag{3.13}$$

The numerical performance of h-p extensions is illustrated in Table 3.8. On comparing the results with those in Table 3.2a, it is seen that the rate of convergence of the h-p extension is much greater than the rate of convergence of the h-extension based on asymptotically optimal meshes.

3.5. The h-p Extension with Uniform p and Optimal Mesh ($\xi = 0$)

The mesh is the same as in Section 3.4, i.e., with the nodal points defined by (3.9), (3.10) and we assume that the p-distribution is uniform. In this case the estimate for $\frac{1}{2} < \alpha < 1$ is:

$$\|e\|_{ER} = C(\alpha) \frac{1}{p^{2\alpha-1}} e^{\frac{-\gamma_0}{\sqrt{2}} \sqrt{(\alpha - 1/2)N}} \tag{3.14}$$

where, as in (3.13), $\gamma_0 = 1.574$ and:

$$p = [s(\alpha)]M(\Delta) \tag{3.15}$$

$$s(\alpha) = 2(\alpha - 1/2). \tag{3.16}$$

The numerical performance of the h-p extension with uniform p is shown in Table 3.9. On comparing Table 3.9 with Table 3.8, it is seen that the performance of the h-p extension with uniform p is not substantially different from that of the h-p extension with optimal p-distribution. The performance can be analyzed also for p-distributions other than that given in (3.15). When p increases more rapidly than (3.15), then the rate of convergence diminishes until it reaches the algebraic rate characteristic of p-extensions. When p increases less than (3.15), then the rate of convergence diminishes because $h(\Delta)$ does not change.

3.6. H-Extensions that Utilize Feedback ($\xi = 0$)

In Section 3.2 it was pointed out that the quality of performance of h-extensions depends on the mesh design. Proper mesh design depends on the exact solution which generally is not known. It is possible to devise feedback procedures however, that construct meshes which asymptotically perform as well as the optimal meshes. Such feedback procedures are called adaptive [9], [10], [11], [12].

Tables 3.10a, 3.10b show the results of numerical experiments. The numerical value D is based on a formula for optimal meshes that utilize only nodal point which can be constructed by successive bisection of elements, not all meshes as considered before, because the feedback procedure uses only such meshes.

3.7. H-p Extensions that Utilize Feedback

It is possible to devise feedback procedures that perform nearly as well as the optimal h-p extension. Results obtained with such a procedure are shown in Table

3.11 ($\alpha = 0.7$). The meshes were generated by bisection, therefore, D is based on an estimate developed for such meshes only.

3.8. p-Extension Based on Properly Designed Meshes and Feedback ($\xi = 0$)

In this case the mesh is strongly graded toward singular points, on the basis of (3.9), with $\kappa_0 = (\sqrt{2} - 1)^2 = 0.1715$ or slightly smaller, say $\kappa_0 = 0.15$ and $M(\Delta)$ is fixed. The polynomial degree of elements is uniformly increased. The error decrease first exponentially and if p is too large, then algebraically, as explained in Sec-tion 3.5. Feedback is utilized to ensure (through proper selection of $M(\Delta)$) that the desired accuracy is reached in the range where the convergence is exponential.

3.9. Smooth Solutions ($\xi > 0$)

We have considered various extension processes when the solution has singular character. When the solution is smooth, then p-extensions perform especially well for small M. The error estimates for $\xi > 0$ in (2.3) are as follows:

(a) for $\xi = 0$

$$\|e\|_{ER} = \frac{C(\alpha)}{p^{2\alpha-1}} \tag{3.17}$$

(b) for $\xi > 0$

$$\|e\|_{E} = C(\alpha)\left(\frac{1-q^2}{2q}\right)\frac{q^p}{p^\alpha} \tag{3.18}$$

where:

$$q = \frac{\sqrt{1+\xi} - \sqrt{\xi}}{\sqrt{1+\xi} + \sqrt{\xi}} \tag{3.19}$$

In Table 2.11 results are presented for $\alpha = 0.7$ and $\xi = 0$, $\xi = 0.01$ and $\xi = 0.1$. These results demonstrate that the performance of p-extensions very rapidly improves with increasing smoothness of the solution.

4. ACCURACY OF STRESS APPROXIMATIONS IN ONE DIMENSION

In one dimension the stress is simply $u'(x_0)$. In contrast to the two dimensional case, the behavior of the finite element solution in our example is entirely local, therefore we need to consider only the case with one element. The results of numerical experiments for $\alpha = 0.7$ and various ξ values are shown in Table 4.1. It is seen that the element that contains the singularity ($\xi = 0$) yields very poor stress approximations.

When quantities other than the energy are of interest, for example stresses, then the mesh and p-distribution should be optimal or nearly optimal with respect to the purpose of computation. Optimal meshes and p-distributions depend not only on the purpose of computation but also on the method used for computing the quantities of interest. See, for example, [10],[13].

5. PERFORMANCE OF THE FINITE ELEMENT METHOD IN TWO DIMENSIONS WITH RESPECT TO THE ENERGY NORM

The theory of two dimensional problems presents more difficulties and is therefore less well understood than the theory of one dimensional problems. Nevertheless there are important similarities which make it possible to gain valuable insight from the analysis of one dimensional problems. There are important differences also, for example, error in stresses in two dimensions behave quite differently from the errors in stresses in one dimension. Stress computations are discussed in the next section. Here we discuss the properties of various versions and present numerical results for our two dimensional model problem. The results were obtained by means of the h-version program FEARS with feedback capabilities [14] and the p-version program FIESTA-2D [15]. FEARS has elements of polynomial degree 1 only, the polynomial degree of elements in FIESTA-2D range from 1 to 8.

5.1. H-Extension Based on Uniform Meshes

The estimate for our model problem, defined in Section 2.2, is:

$$\|e\|_{ER} = C(p)N^{-1/4}. \tag{5.1}$$

Detailed theoretical analysis, comparable to the one dimensional case, is not available. The results of computations are shown in Table 5.1, where D represents the numerical value of C(p). The results indicate that the asymptotic estimate (5.1) is of good quality.

5.2. H-Extension with Feedback

As in the one dimensional case, the sequence of optimally designed meshes leads to a rate of convergence independent of the singularity. The estimate for optimally designed meshes is:

$$\|e\|_{ER} = C(p)N^{-p/2} \tag{5.2}$$

Note that the exponent of N is p/2, not p as in the one dimensional case.

H-extension with properly utilized feedback (adaptive approach) should lead to the same asymptotic rate of convergence as the optimally designed sequence of meshes. Table 5.2 shows the results obtained with FEARS.

5.3. P-Extension on Uniform Mesh

In this case the estimate is [3] [4]:

$$|e|_{ER} = C(\varepsilon)N^{-1/2+\varepsilon} \tag{5.3}$$

where $\varepsilon > 0$, arbitrary. It is not known whether the term ε can be removed.

Table 5.3 shows the results obtained with FIESTA-2D using four square elements.

5.4 H-p Extension

The estimate for h-p extensions based optimal mesh and on either optimally or uniformly distributed p is:

$$\|e\|_{ER} \leq C\, e^{-\gamma\sqrt[3]{N}}, \qquad \gamma > 0. \tag{5.4}$$

In the two dimensional case the optimal value of γ is not known nor is it known whether the term $\sqrt[3]{N}$ can be improved. The value of γ depends on the distribution of p.

5.5. p-Extension Based on Properly Designed Mesh and Feedback Information

As in the one dimensional case, the p-extension performs much the same way as the h-p extension when p is not two large and the mesh is properly designed. For large p the p-version performs as if the mesh were uniform. An example is presented in Table 5.4. The fact that the rate of convergence slows for high p is an indication that the mesh should be refined. Slowing of the rate of convergence can be detected which is the feedback information needed for increasing the number of finite elements.

5.6. Smooth Solutions

When the solution is smooth, the p-version is very effective and, as in the one dimensional case, the convergence is exponential.

6. STRESS COMPUTATIONS IN TWO DIMENSIONS

Stress approximations behave quite differently in two dimensions than in one dimension. In one dimension in our example the error depended only on the behavior of the solution in the particular element in question, i.e., the error was completely localized. In two dimensions, on the other hand, the error is comprised of two parts: the local error and the error associated with all other elements. This second part is called pollution error.

The error in stresses depends to a large extent on how the stresses are computed. Indirect techniques are available which substantially reduce both the local and pollution error, as compared with the conventional (direct) methods of stress computations [7], [10], [13].

6.1. Performance of h-Extensions Based on Nearly Optimal Meshes

Let us examine the stresses at point $x_1 = x_2 = 0.25$ in the model problem. Solutions were obtained by means of the computer program FEARS. The point under consideration is the vertex of four elements. Therefore, four different values can be computed, using the derivatives of the four elements and the appropriate stress-strain low. The relative errors are shown in Table 6.1 for the three stress components σ_{11}, σ_{22}, σ_{12} for the four adjacent elements. The error of the average value (A) is also shown. It is seen that the error of the average value is smaller than the error in most elements. This is a well known fact which is generally utilized in stress computations.

The relative error in the same stress componets computed by means of an indirect (postprocessing) technique [13] is shown in Tab. 6.2.

The improvement is very substantial. The postprocessing technique yields stress values which are not sensitive to the meshes and the error is of the magnitud $\|e\|^2_{ER}$.

6.2. Performance of the p-Version

When the solution is smooth, the p-version performs well. When the solution is not smooth and the elements are large, then the pollution error is generally large. Satisfactory theoretical analysis is not available. It is known, however, that the pollution error can be reduced very substantially by surrounding points of stress singularity with one or more layers of elements.

The relative errors at point $x_1 = 0.1$, $x_2 = 0.2$ (which is located at element boundaries) are shown in Tables 6.3a and 6.3b. The results presented in Table 6.3a are strongly affected by pollution because the vertex of the neighboring element was on the singular point. The results presented in Table 6.3b are much less affected by pollution because an extra layer of elements were added so that the neighboring element no longer had a vertex on the singular point. The local error is, of course, the same in both cases. Table 6.3a and 6.3b illustrate the importance of proper mesh design when the stresses are computed from the finite element solution directly. The postprocessing method removes sensitivity to mesh design.

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TABLE 3.1.- H-EXTENSION WITH UNIFORM MESH REFINEMENT, α = .7

M	p - 1		p = 2		p = 3		p = 4	
	$\|e\|_{ER}\%$	D	$\|e\|_{ER}\%$	D	$\|e\|_{ER}\%$	D	$\|e\|_{ER}\%$	D
2	87.31	1.00	66.57	1.01	56.70	1.01	50.58	1.01
4	76.11	1.00	57.96	1.01	49.37	1.01	44.03	1.01
8	66.26	1.00	50.44	1.01	42.98	1.01	38.33	1.01
16	57.70	1.00	43.91	1.01	37.40	1.01	33.36	1.01
32	50.23	1.00	38.24	1.01	32.57	1.00	29.05	1.01
64	43.72	1.00	33.29	1.01	28.35	1.00	25.29	1.01
128	38.08	1.00	28.98	1.00	24.68	1.01	22.01	1.01
256	33.13	1.01	25.22	1.01	21.48	1.00	19.16	1.01

TABLE 3.2a.- H-EXTENSION WITH ASYMPTOTICALLY OPTIMAL MESH, α = 0.7

M	p = 1 (β = 7.50)		p = 2 (β = 12.50)	
	$\|e\|_{ER}\%$	D	$\|e\|_{ER}\%$	D
2	80.71	.314	70.65	.133
4	51.59	.402	39.94	.300
8	29.14	.454	14.77	.445
16	15.46	.482	4.365	.526
32	7.966	.496	1.168	.562

TABLE 3.2b.- H-EXTENSION WITH ASYMPTOTICALLY OPTIMAL MESH, α = 1.1

M	p = 1 (β =2.50)		p = 2 (β=3.17)		p = 3 (β=5.83)		p = 4 (β=7.50)	
	$\|e\|_{ER}\%$	D	$\|e\|_{ER}\%$	D	$\|e\|_{ER}\%$	D	$\|e\|_{ER}\%$	D
2	58.91	1.19	21.99	.851	15.10	.540	12.66	.275
4	31.88	1.29	7.128	1.20	3.162	.905	1.982	.688
8	16.55	1.34	2.042	1.23	.4953	1.13	.1812	1.01
16	8.434	1.37	.5387	1.33	.688(-1)	1.26	.134(-1)	1.19
32	4.257	1.41	.1361	1.34	.904(-1)	1.32	.908(-3)	1.29
64	2.138	1.39	.345(-1)	1.36	.115(-2)	1.35	.586(-4)	1.33
128	1.071	1.38	.868(-2)	1.38	.146(-3)	1.38	.376(-5)	1.37
256	.5364	1.38	.217(-2)	1.38	.184(-4)	1.38	.240(-6)	1.38

TABLE 3.3.- PERFORMANCE OF AN OPTIMAL MESH ($M = 2$, $\alpha = 1.1$)

$p = 1$	$p = 2$	$p = 3$	$p = 4$
$\|e\|_{ER}\%$	$\|e\|_{ER}\%$	$\|e\|_{ER}\%$	$\|e\|_{ER}\%$
57.42	17.31	8.213	4.762

TABLE 3.4.- MESH PARAMETERS ($\alpha = 0.7$)

p	β_0	$\bar{\beta}$	x_1^{Δ}(opt)	x_1^{Δ}(min)
1	7.5	5	5.14(-12)	2.99(-8)
2	12.50	10	1.53(-19)	8.88(-16)
3	17.50	15	4.56(-27)	2.64(-23)
4	22.50	20	1.36(-36)	7.88(-31)
5	27.50	25	4.05(-42)	2.35(-38)

TABLE 3.5.- PENALTY FOR USING OVERREFINED MESH IN TERMS OF INCREASED VALUES OF C (FIRST ROW) AND N (SECOND ROW)

α / p	.7	1.1	1.6	2.1
1	1.00 1.00	1.97 1.97	3.40 3.40	4.81 4.81
2	1.00 1.00	4.69 2.17	14.73 3.84	29.97 5.47
3	1.00 1.00	12.08 2.29	69.09 4.10	202.90 5.88
4	1.00 1.00	32.19 2.30	337.15 4.29	1430.90 6.15

TABLE 3.6.- P-EXTENSIONS ON UNIFORM MESHES (α = .7)

P	M = 1		M = 2		M = 3		M = 4	
	$\|e\|_{ER}\%$	D	$\|e\|_{ER}\%$	D	$\|e\|_{ER}\%$	D	$\|e\|_{ER}\%$	D
1			87.31	1.00	80.59	1.00	76.11	1.00
2	76.47	1.01	66.57	1.01	61.39	1.01	57.96	1.01
3	65.14	1.01	56.70	1.01	52.29	1.01	49.37	1.01
4	58.10	1.01	50.58	1.01	46.64	1.01	44.03	1.01
5	53.15	1.01	46.27	1.01	42.67	1.01	40.27	1.01
6	49.42	1.01	43.02	1.01	39.66	1.01	37.45	1.01
7	46.48	1.01	40.46	1.01	37.31	1.01	35.21	1.01
8	44.05	1.01	38.36	1.01	35.37	1.01	33.39	1.01
9	42.02	1.01	36.58	1.01	33.74	1.01	31.85	1.01
10	40.29	1.01	35.09	1.01	32.34	1.01	30.54	1.01
11	38.80	1.01	33.76	1.01	31.15	1.01	29.40	1.01

TABLE 3.7.- PERFORMANCE OF p-EXTENSION BASED ON OPTIMAL p-DISTRIBUTION AND UNIFORM MESH (M = 2)

α	p_1	p_2	$\|e\|_{ER}\%$
.7	41	1	5.05
1.1	3	1	23.79
1.1	26	2	1.834

3.8.- PERFORMANCE OF h-p EXTENSION (α= 0.7, s = 0.4), ASYMPTOTICALLY OPTIMAL MESH AND p-DISTRIBUTION

M	N	$\|e\|_{ER}\%$	D	M	N	$\|e\|_{ER}\%$	D
2	3	54.18	2.62	8	21	7.031	2.82
3	5	38.68	2.66	9	25	4.998	2.77
4	7	28.02	2.61	10	30	3.523	2.82
5	10	19.82	2.70	11	35	2.492	2.80
6	13	14.10	2.69	12	41	1.754	2.89
7	17	9.942	2.82	20	100	.1071	2.94

TABLE 3.9.- PERFORMANCE OF THE h-p EXTENSION, ASYMPTOTICALLY OPTIMAL MESH AND UNIFORM p (α= .7)

M	N	p	$\|e\|_{ER}\%$	D
2	1	1	74.36	1.50
3	5	2	38.68	1.70
4	7	2	28.02	1.50
5	14	3	16.17	1.57
8	31	4	5.037	1.46
10	49	5	2.262	1.43
15	104	7	.3376	1.25

TABLE 3.10a.- H-EXTENSION UTILIZING FEEDBACK, α = 0.7

p = 1			p = 2		
N	$\|e\|_{ER}\%$	D	N	$\|e\|_{ER}\%$	D
4	58.54	.570	9	43.93	.516
9	31.66	.617	19	22.01	1.03
20	14.48	.592	29	11.09	1.17
39	7.326	.571	39	5.719	1.07
85	3.327	.557	81	1.121	.886
101	2.788	.553	101	.3689	.802

TABLE 3.10b.- H-EXTENSION UTILIZING FEEDBACK, α = 1.1

p - 1			p = 2			p = 3		
N	$\|e\|_{ER}\%$	D	N	$\|e\|_{ER}\%$	D	N	$\|e\|_{ER}\%$	D
4	28.21	1.44	9	8.295	2.01	14	4.979	2.78
11	12.06	1.46	21	1.567	1.83	29	.6506	2.91
19	7.255	1.46	41	.4123	1.76	44	.1736	2.62
28	5.011	1.47	51	.2710	1.77	62	.05706	2.36
37	3.837	1.47	79	.1111	1.72	77	.02979	2.34
63	2.266	1.45	99	.07155	1.73	95	.01617	2.37
137	1.052	1.46	125	.04411	1.69	137	.005006	2.17
255	.5673	1.46	251	.01093	1.68	227	.001095	2.15

TABLE 3.11.- H-p EXTENSION UTILIZING FEEDBACK, $\alpha = 0.7$

N	$\|e\|_{ER}\%$	D
4	58.54	2.795
9	31.66	2.88
25	10.47	3.69
39	4.235	3.52
60	1.548	3.64
120	.2933	3.51

TABLE 3.12.- PERFORMANCE OF THE p-EXTENSION WITH $M = 1$, $\alpha = 0.7$ and $\xi = 0, 0.1, 0.1$

	$\xi = 0$		$\varepsilon = .01$		$\xi = .1$	
P	$\|e\|_{ER}\%$	D	$\|e\|_{ER}\%$	D	$\|e\|_{ER}\%$	D
2	76.47	1.01	57.75	.391	36.61	.443
3	65.14	1.01	37.90	.416	15.42	.462
4	58.10	1.01	26.37	.432	6.922	.473
5	53.15	1.01	18.57	.443	3.224	.481
6	49.42	1.01	13.91	.451	1.541	.486
7	46.47	1.01	10.38	.458	.7578	.490
8	44.05	1.01	7.831	.463	.3677	.494
9	42.01	1.01	5.961	.468	.1825	.495
10	40.29	1.01	4.571	.471	.09136	.496
11	38.80	1.01	3.525	.475	.04603	.498

TABLE 4.1.- ERROR IN STRESSES IN DEPENDENCE ON POLYNOMIAL DEGREE AND SMOOTHNESS

ξ	x_0 \ p	2	3	4	5	6	7	8	9	10	11	EXACT
0	0	/	/	/	/	/	/	/	/	/	/	∞
	.25	.2910	.2151	- .0159	-.1538	-.1143	.0186	.1052	.0780	.0165	-.0803	.0871
	.50	.1974	- .1060	- .1060	-.0728	-.0728	-.0554	-.0557	.0452	.0452	-.0382	-.1974
	.75	- .0397	- .1156	- .1155	-.0223	-.0618	.0710	-.0156	-.0427	.0518	-.0120	-.3384
	1.0	- .3277	.2792	- .2490	.2278	-.2118	.1992	-.1888	.1801	-.1727	.1663	-.4286
.01	0	-1.960	-1.519	-1.193	-.9440	-.7511	-.5999	-.4805	-.3858	-.3103	-.2500	.2599
	.25	.2036	.1485	.0057	-.0661	-.0488	.0001	.0267	.0198	-.0005	-.0118	.1163
	.50	.1578	- .0625	- .0625	-.0307	-.0307	-.0165	-.0165	.0094	.0094	.0055	-.1578
	.75	- .0240	- .0791	- 0.636	-.0083	-.0256	.0232	-.0034	-.0104	.0098	-.0015	-.2959
	1.0	- .2551	.1857	- .1404	.1083	-.0846	.0666	-.0528	.0419	-.0335	.0268	-.3827
.1	0	- .3641	- .1809	- .0922	-.0476	-.0246	-.0129	-.0067	-.0035	-.0019	-.0010	.7532
	.25	- .0678	- .0452	.0064	-.0065	-.0044	-.0006	.0007	.0005	.0001	.0001	.1281
	.50	- .0765	- .0138	.0138	-.0030	-.0030	-.0007	-.0007	.0002	.0002	.0000	-.0765
	.75	- .0037	- .0263	.0125	-.0005	-.0025	.0013	-.0001	-.0003	.0002	.0000	-.1921
	1.0	- .1215	.0590	- .0296	.0151	-.0078	.0040	-.0021	.0011	-.0006	.0003	-.2703

TABLE 5.1.- H-EXTENSION BASED ON UNIFORM MESHES (p = 1)

N	$\|e\|_{ER}\%$	D
51	36.02	.96
167	27.07	.97
591	19.81	.97

TABLE 5.2.- H-EXTENSION WITH FEEDBACK (p = 1)

N	$\|e\|_{ER}\%$	D
67	32.91	2.03
101	26.38	2.66
143	21.35	2.56
221	16.79	2.50
301	13.61	2.36
617	9.63	2.40

TABLE 5.3.- PERFORMANCE OF THE p-EXTENSION (UNIFORM MESH, 4 ELEMENTS)

p	$\|e\|_{ER}\%$	D
1	32.61	2.01
2	18.35	1.82
3	15.89	1.99
4	13.24	2.06
5	11.06	2.06
6	9.47	2.07
7	8.27	2.08
8	7.37	2.08

TABLE 5.4.- P-EXTENSION BASED ON PROPERLY DESIGNED MESH

P	$\|e\|_{ER}$%
1	31.96
2	12.36
3	6.197
4	3.277
5	2.131
6	1.436
7	1.128
8	.8839

TABLE 6.1.- RELATIVE ERROR IN STRESS COMPONENTS AT POINT (0.25, 0.25) DIRECT COMPUTATION

N		e^R_{11}%	e^R_{22}%	e^R_{12}%
221	A	10.99	7.57	11.16
	1	4.79	2.32	35.86
	2	12.21	2.01	.093
	3	17.27	17.42	106.43
	4	9.71	13.01	71.49
617	A	4.09	5.47	13.42
	1	1.46	2.68	22.96
	2	4.41	.099	55.69
	3	4.41	13.43	3.88
	4	6.07	11.74	28.85

TABLE 6.2.- RELATIVE ERROR IN THE STRESS COMPONENTS AT POINT (0.25, 0.25) INDIRECT COMPUTATION

N	e^R_{11}%	e^R_{22}%	e^R_{12}%
221	1.69	2.63	1.94
617	.56	.866	.81

TABLE 6.3a.- RELATIVE ERROR IN STRESSES. A VERTEX OF THE ADJACENT ELEMENT LIES ON THE CRACK TIP

p	$e^R_{11}\%$	$e^R_{22}\%$	$e^R_{12}\%$
1	7.903	34.17	18.18
2	6.222	14.09	11.20
3	1.014	10.53	3.254
4	5.249	9.652	4.844
5	2.864	5.411	.1114
6	.6259	3.848	.8387
7	.6893	3.119	.9926
8	1.438	2.233	1.316

TABLE 6.3b.- RELATIVE ERROR IN STRESSES. THE ADJACENT ELEMENT IS SEPARATED FROM THE CRACK TIP BY ONE LAYER OF ELEMENTS

p	$e^R_{11}\%$	$e^R_{22}\%$	$e^R_{12}\%$
1	13.57	33.108	17.39
2	2.124	6.976	.6688
3	1.091	3.843	3.620
4	.9997	1.923	1.852
5	.2653	.7836	.7166
6	.1702	.3928	.3503
7	.0784	.2123	.1529
8	.04005	.1256	.07431

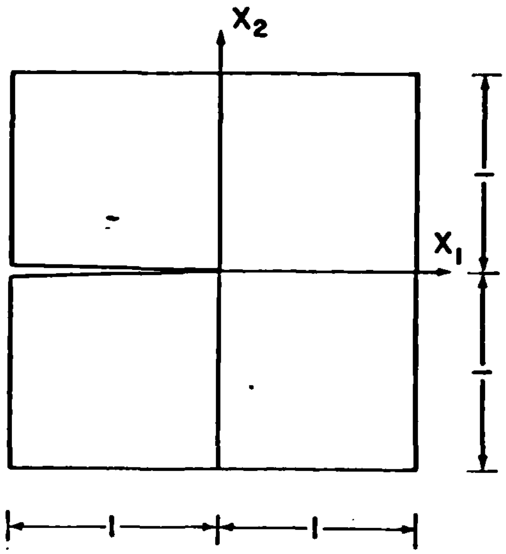

Figure 2.1. Scheme of the cracked panel.